Polyphase Barker Sequences up to Length 31

Abstract: Polyphase sequences are finite, complex, time-discrete sequences with constant magnitude and variable phases. A polyphase sequence with elements of magnitude 1 is called a Barker sequence if the maximum magnitude of all sidelobes of their aperiodic autocorrelation function (ACF) is less or equal to one. Stochastic optimization algorithms have been applied to search for polyphase Barker sequences. New sequences meeting the Barker condition up to length 31 have been found.

Introduction: Polyphase sequences with low autocorrelation sidelobes are applied for example in radar and system identification. Because of the constant magnitude they have a valuable feature: With a given peak power constraint on the transmitter their energy is higher than that of any other non-constant magnitude sequence. The low magnitude of the ACF sidelobes in case of Barker sequences ensures an easily detectable peak at the output of a matched filter receiver.

Binary Barker sequences with elements $a_n \in \{-1, +1\}$ are only known up to length 13 [1]. Increasing the size of the alphabet allows the construction of longer Barker sequences. M - Polyphase sequences have elements $\underline{a}_n \in \{\exp(2\pi i/M)\}$ with $i = 0 \ldots M-1$. They can be regarded as a generalization of binary sequences which are included as a special case with $M = 2$. When M gets infinite, the phases become continuous variables and the sequence is called uniform. This paper refers to polyphase sequences with large M.

Given a complex sequence $\{\underline{a}_n\}$ with $n = 0 \ldots N-1$ their aperiodic autocorrelation function $\{\underline{r}_k\}$, is defined as:

$$\underline{r}_k = \sum_{n=0}^{N-k} \underline{a}_n^* \cdot \underline{a}_{n+k} \qquad k = -(N-1) \ldots + (N-1) \tag{1}$$

For sequences with constant magnitude $|\underline{a}_n| = 1$ being investigated only, the sequence $\{\underline{a}_n\}$ can be represented by a sequence of phases $\{\varphi_n\}$. Note, that in this case because of Eq.(1): $|\underline{r}_{N-1}| = |\underline{r}_{-(N-1)}| = 1$. Sequences with an ACF that has low sidelobes are of particular interest. Minimizing a cost function:

$$C(p) = \sqrt[p]{\sum_{k=1}^{N-1} |\underline{r}_k|^p} \qquad p > 0 \tag{2}$$

It is sufficient to evaluate only ACF-lags with positive index k, because $|\underline{r}_k| = |-\underline{r}_k|$. In case of $p = 2$ the total energy of the sidelobe lags is minimized. In case of $p = \infty$ the maximum magnitude of all sidelobe lags is minimized. A sequence is called a Barker sequence if $C(\infty) \leq 1$.

Optimization Algorithm: In order to discover polyphase sequences with low ACF sidelobes, the heuristic Great Deluge Algorithm (GDA) has been used. It had been applied successfully to the traveling salesman problem with many parameters [2] and has outperformed other stochastic optimization algorithms like Simulated Annealing (SA) or Threshold Accepting (TA). Figure 1 shows the flowchart of its adaption to the ACF sidelobe minimization problem.

- Choose initial phase configuration vector $\{\varphi_n\}$ with $0 \leq \varphi_n \leq 2\pi$
- Choose initial phase stepsize vector $\{\Delta\varphi_n\}$ with $0 < \Delta\varphi_n \leq \pi$
- Choose minimum phase stepsize $MP > 0$
- Choose phase divisor $PD > 1$
- Choose "rain speed" $RS > 0$
- $WL = Max| - C(p)|$ ("water level")
- $UA = 0$ (unsuccessful alterations)
- WHILE $|\Delta\varphi_n| > MP \quad \forall n$
  - $DS = 0$ (dry steps)
  - FOR $n = 1$ TO $n = N$
    - $\varphi_n = \varphi_n + \Delta\varphi_n$
    - $Q = Max| - C(p)|$
    - IF $Q < WL$
    - THEN
      - $\Delta\varphi_n = -\Delta\varphi_n$ (invert phase step direction)
      - $\varphi_n = \varphi_n + \Delta\varphi_n$ (go back to old position)
    - ELSE
      - $WL = WL + RS$ (increase "water level")
      - $DS = DS + 1$
  - IF $DS = 0$ (if no alteration could improve Q)
  - THEN
    - $UA = UA + 1$
    - IF $UA = 2$ (no improvement possible with actual phase stepsize)
    - THEN
      - FOR $n = 1$ TO $n = N$
        - $\Delta\varphi_n = \Delta\varphi_n / PD$ (decrease phase stepsize)
      - $UA = 0$
  - ELSE
    - $UA = 0$

Figure 1: GDA for optimization of ACF

The principle of the GDA is very simple. Assume a given quality function Q as a function of N parameters. A maximum of Q is searched for, the global maximum if possible. In the case of $N = 2$, Q may be interpreted as a hilly terrain with mountains and valleys. During the optimization process the statistically varying set of optimization parameters describes a movement in this area reaching higher and lower points on the Q-surface. A simple gradient algorithm can only move upwards. In contrast to that, the GDA allows every movement that does not result in a Q lower than a certain threshold, the "water level". During optimization this "water level" is continously increased. This fact has given the algorithm its name. The algorithm forces the movement into areas with a Q of higher quality, while leaving enough space to get away of suboptimum local optima.

In our calculations the creation of the new phase sequence $\{\varphi_n\}$ is rather deterministic. Each phase value φ_n is altered according to the following rule. First, φ_n is increased by $\Delta\varphi_n$. If the resulting Q is accepted, the next phase is altered, if not, φ_n is decreased by $\Delta\varphi_n$. If no alteration of any phase is accepted, the phase stepsize is decreased. The GDA is terminated when the phase stepsize becomes smaller than a predefined minimum phase stepsize.

N	20	26	27	28	29	30	31
$C(\infty)$	0.979480	0.879470	0.985172	0.950203	0.871080	0.997476	0.934462
$\{\varphi_n\}$	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
	4.553049	0.895841	0.184756	5.463798	0.120401	0.577184	0.495357
	4.086800	2.043260	0.381913	4.811496	5.552993	0.603685	2.054263
	2.215325	2.412575	0.501476	3.381014	4.186235	0.587563	2.880816
	1.894461	4.632634	5.667001	3.739787	4.619877	0.207471	4.126907
	3.945793	4.659854	5.383231	2.796088	4.174523	5.237221	5.387150
	3.770581	3.061830	4.897648	4.448612	2.799691	4.913640	5.322754
	0.092210	2.055563	2.065845	4.248550	5.262097	0.461842	4.127543
	0.967971	4.540587	1.731439	5.258279	5.715868	0.945976	3.776533
	2.931040	3.491350	1.958458	4.368063	0.326038	2.715992	5.713595
	1.003241	2.376147	4.965856	0.595450	5.580307	3.698029	4.877785
	2.452194	2.690606	3.500713	5.859119	1.481594	4.041807	3.687515
	3.460843	3.123204	5.477305	2.969228	1.894634	2.346154	4.313581
	1.330362	1.322308	2.029632	5.908823	3.910927	1.326526	3.350924
	3.745265	5.951858	5.701172	2.858973	0.109814	5.544522	1.665117
	1.916620	3.270461	3.226034	5.257347	0.547566	4.813630	0.296297
	4.431156	5.356688	0.931898	0.584991	3.212265	1.179540	4.764131
	2.297439	3.393189	0.153375	4.028262	2.928542	5.217885	0.920844
	4.416524	1.614152	3.384669	1.763293	1.569419	3.222122	5.779411
		3.319561	1.694949	0.929504	5.675662	1.267677	3.908770
		0.300302	4.205310	4.128530	3.971280	2.683552	5.301284
		1.920568	5.851580	4.342097	2.537828	0.115401	2.568618
		4.368801	1.796881	0.824984	5.757173	4.583854	0.379536
		0.674586	3.989645	1.073402	1.598000	1.642393	4.286983
		3.485233	5.794346	3.249433	4.602239	4.237597	0.511796
			1.622635	4.575381	1.641441	6.267467	2.539604
				0.561518	4.413265	2.612215	5.184811
					1.039439	5.346838	1.089083
						1.248448	3.330664
							0.135490

Table 1: Phases of new uniform Barker sequences

Results: Table 1 shows the phases of the new polyphase Barker sequences of length $N = 26$ to 31. Shorter polyphase Barker sequences are available in [3] and are not listed for that reason. Table 1 also provides the phases of a sequence of length 20, because in [3] the sequence of length 20 does not meet the barker condition. For every length N the lexicographically smallest representative of its equivalence class [4] is shown. This means, that $\varphi_0 = \varphi_1 = 0$.

We observed that it is quite simple to find a good local optimum with the GDA, but to discover the global optimum probably depends on a lucky choice of the initial phase configuration. This assumption is enforced by the observation, that after several runs of the algorithm with different initial phases at lower N many different Barker sequences were found. Opposed to that, with $N = 31$ it took several runs to find a Barker sequence at all. After some experiments we found out that it is useful to proceed in two steps. In a first step $C(2)$ is minimized. After several program runs with different starting values sometimes a reasonably good result was obtained. For these cases $C(\infty)$ has been minimized in a second step. Using this procedure, sequences of length 32 with $C(\infty) < 1.22$ have been found. These results give rise to the hope that Barker sequences also exist for $N > 31$ and that they might be found with somewhat more ingenious algorithms.

M. FRIESE
H. ZOTTMANN

19th September 1994

Institute for Communications and Electroacoustics
Technical University of Darmstadt
Merckstraße 25, 64283 Darmstadt, Germany

Acknowledgements: We would like to thank the “Forschungs- und Technologiezentrum (FTZ)” of the “Deutsche Bundespost Telekom, Darmstadt” for financial support.

References

[1] Hans Dieter Lüke: Korrelationssignale. Springer Verlag, 1992

[2] Gunter Dueck: New Optimization Heuristics, The Great Deluge Algorithm and the Record-to-Record Travel. *JOURNAL OF COMPUTATIONAL PHYSICS*, Vol. 104, 1993, pp. 86 - 92

[3] L. Bömer and M. Antweiler: Polyphase Barker Sequences. *ELECTRONICS LETTERS*, Vol. 25, No. 23, Nov. 1989, pp. 1577 - 1579

[4] S.W. Golomb and R.A. Scholz: Generalized Barker Sequences. *IEEE Transactions on Information Theory*, Vol. IT-11, No.4, Oct. 1965, pp. 533 - 537